

# FRACTIONAL HARDY-TYPE INEQUALITIES IN DOMAINS WITH PLUMP COMPLEMENT

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ABSTRACT. We establish fractional Hardy-type inequalities in a bounded domain with plump complement. In particular our results apply in bounded  $C^\infty$  domains and Lipschitz domains.

## 1. INTRODUCTION

Let  $\Omega$  be a proper subdomain in  $\mathbb{R}^n$ ,  $n \geq 2$ . Let  $s \in (0, 1)$  and let  $p, q \in (1, \infty)$  be given such that  $0 < 1/p - 1/q < s/n$ . We investigate the inequality

$$(1.1) \quad \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial\Omega)^{q(s+n(1/q-1/p))}} dx \leq c \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{ps+n}} dy dx \right)^{q/p}$$

for every  $u \in W^{s,p}(\mathbb{R}^n)$  with  $\text{spt } u \subset \overline{\Omega}$ ; here the finite constant  $c$  depends only on  $s, n, p, q, \Omega$ . Our work was motivated by the following fractional order inequality

$$(1.2) \quad \int_{\Omega} \frac{|u(x)|^p}{\text{dist}(x, \partial\Omega)^{ps}} dx \leq c \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{ps+n}} dy dx$$

for all  $u \in C_0(\Omega)$  with a finite constant  $c$  which depends only on  $s, n, p$ , and  $\Omega$ . B. Dyda proved that inequality (1.2) holds in  $\Omega$  with  $p > 0$ , if one of the following conditions is valid:

- (1) if  $\Omega$  is a bounded Lipschitz domain and  $sp > 1$ ,
- (2) if  $\Omega$  is a complement of a bounded Lipschitz domain and  $sp \in (0, \infty) \setminus \{1, n\}$ ,
- (3) if  $\Omega$  is a complement of a point and  $sp \in (0, \infty) \setminus \{n\}$ ,

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- (4) if  $\Omega$  is a domain above the graph of a Lipschitz function  $\mathbb{R}^{n-1} \rightarrow \mathbb{R}$  and  $sp \in (0, \infty) \setminus \{1\}$ ,

[D, Theorem 1.1]. He showed also that inequality (1.2) is false if  $\Omega$  is a bounded Lipschitz domain with  $sp \leq 1$  and  $s < 1$ . Inequality (1.2) was proved for convex domains when  $1 < p < \infty$  and  $1/p < s < 1$  by M. Loss and C. A. Sloane, [LS, Theorem 1.2]. Inequality (1.2) holds in a half-space whenever  $0 < s < 1$ ,  $sp \neq 1$ ,  $1 \leq p < \infty$ , by R. L. Frank and R. Seiringer [FS, Theorem 1.1]; the  $p = 2$ -case was considered in [BD, Theorem 1.1].

We prove fractional Hardy-type inequalities (1.1) in a bounded domain whose complement is plump in the sense of the following definition. The open and closed  $n$ -dimensional Euclidean balls, centered at a point  $x$  and with radius  $r > 0$ , are denoted by  $B^n(x, r)$  and  $\overline{B^n(x, r)}$ , respectively.

**1.3. Definition.** Let  $n \geq 2$  and  $\eta \geq 1$ . A set  $A$  in  $\mathbb{R}^n$  is  $\eta$ -plump if for all  $x \in \bar{A}$  and all  $r \in (0, \text{diam}(A))$  there is a point  $z$  in  $\overline{B^n(x, r)}$  with  $B^n(z, r/\eta) \subset A$ .

The following is our main theorem.

**1.4. Theorem.** Suppose that  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , with an  $\eta$ -plump complement  $\mathbb{R}^n \setminus \Omega$ ,  $\eta \geq 1$ . Let  $s \in (0, 1)$  and let  $p, q \in (1, \infty)$ . If  $0 < 1/p - 1/q < s/n$ , then

$$\begin{aligned} & \left( \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial\Omega)^{q(s+n(1/q-1/p))}} dx \right)^{1/q} \\ & \leq c_{s,n,p,q} \eta^{2n/q+s-n/p} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{sp+n}} dy dx \right)^{1/p} \end{aligned}$$

for every  $u \in W^{s,p}(\mathbb{R}^n)$  with  $\text{spt } u \subset \overline{\Omega}$ .

Examples of bounded domains with  $\eta$ -plump complement include Lipschitz domains and convex domains. More examples are obtained by using  $K$ -quasiconformal mappings  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ : if  $\Omega$  in  $\mathbb{R}^n$  is a bounded domain with an  $\eta$ -plump complement, then the image  $f\Omega$  is also bounded and has a  $\mu$ -plump complement, where  $\mu$  depends on  $n, K$  and  $\eta$  only, see e.g. [V, Theorem 6.6].

We give applications of Theorem 1.4 in Section 4.

## 2. NOTATION AND AUXILIARY RESULTS

The Lebesgue measure of a measurable set  $E$  in  $\mathbb{R}^n$  is written as  $|E|$ . For a measurable set  $E$ , with a finite and positive measure, we write

$$\oint_E f(x) dx = \frac{1}{|E|} \int_E f(x) dx.$$

We write  $\chi_E$  for the characteristic function of a set  $E$ .

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and let  $\mathcal{W}$  be its Whitney decomposition. For the properties of Whitney cubes  $Q \in \mathcal{W}$  we refer to E. M. Stein's book, [S]. In particular, we need the inequalities

$$(2.1) \quad \text{diam}(Q) \leq \text{dist}(Q, \partial\Omega) \leq 4 \text{diam}(Q), \quad Q \in \mathcal{W}.$$

We let  $Q \in \mathcal{W}$  be a cube with center  $x_Q$  and side length  $\ell(Q)$ . By  $tQ$ ,  $t > 0$ , we mean a cube with sides parallel to those of  $Q$  that is centered at  $x_Q$  and whose side length is  $t\ell(Q)$ .

We recall definition of the *fractional order Sobolev spaces* in a domain  $\Omega$  in  $\mathbb{R}^n$ . For  $1 \leq p < \infty$  and  $s \in (0, 1)$  we let  $W^{s,p}(\Omega)$  be the collection of all functions  $f$  in  $L^p(\Omega)$  with  $\|f\|_{W^{s,p}(\Omega)} := \|f\|_{L^p(\Omega)} + |f|_{W^{s,p}(\Omega)} < \infty$ , where

$$|f|_{W^{s,p}(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{sp+n}} dx dy \right)^{1/p}.$$

The support of a function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is denoted by  $\text{spt } f$ , and it is the closure of the set  $\{x : f(x) \neq 0\}$  in  $\mathbb{R}^n$ .

The notation  $a \lesssim b$  mean that an inequality  $a \leq cb$  holds for some constant  $c > 0$  whose exact value is not important. We use subscripts to indicate the dependence on parameters, for example, a quantity  $c_d$  depends on a parameter  $d$ .

We state fractional Sobolev–Poincaré inequalities for a cube.

**2.2. Lemma.** *Let  $Q$  be a cube in  $\mathbb{R}^n$ ,  $n \geq 2$ . Suppose that  $p, q \in [1, \infty)$ , and  $s \in (0, 1)$  satisfy  $0 \leq 1/p - 1/q < s/n$ . Then, for every  $u \in L^p(Q)$ ,*

$$\frac{1}{|Q|} \int_Q |u(x) - u_Q|^q dx \leq c |Q|^{qs/n - q/p} \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dy dx \right)^{q/p}.$$

Here the constant  $c > 0$  is independent of  $Q$  and  $u$ .

*Proof.* The inequality follows from [H-SV, Remark 4.14], when  $Q = [-1/2, 1/2]^n$ . A change of variables gives the general case.  $\square$

Let  $0 < \sigma < d$ . The Riesz potential of a function  $f$  is given by

$$I_{\sigma} f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-\sigma}} dy.$$

The following theorem is from [He, Theorem 1].

**2.3. Theorem.** *Suppose that  $0 < \sigma < d$  and let  $p, q \in (1, \infty)$ . If*

$$0 < 1/p - 1/q = \sigma/d,$$

*then there is a constant  $c > 0$  such that inequality  $\|I_\sigma(f)\|_q \leq c\|f\|_p$  holds for every  $f \in L^p(\mathbb{R}^d)$ .*

Recall from [A] that the fractional maximal function of a locally integrable function  $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$  is

$$\mathcal{M}_\sigma f(x) = \sup_{r>0} \frac{r^\sigma}{|B^d(x, r)|} \int_{B^d(x, r)} |f(y)| dy.$$

If  $Q$  is a cube in  $\mathbb{R}^d$  and  $x \in Q$ , then

$$(2.4) \quad \frac{\ell(Q)^\sigma}{|Q|} \int_Q |f(y)| dy \leq c_d \mathcal{M}_\sigma f(x).$$

Since  $0 < \sigma < d$ , there is a constant  $c_d > 0$  such that

$$(2.5) \quad \mathcal{M}_\sigma f(x) \leq c_d I_\sigma |f|(x)$$

for every  $x \in \mathbb{R}^d$ .

**2.6. Lemma.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and let  $\mathcal{W}$  be its Whitney decomposition. Suppose that  $1 < r < p < q < \infty$  and  $\kappa \geq 1$ . Then*

$$(2.7) \quad \sum_{Q \in \mathcal{W}} |\kappa Q|^{2\beta} \left( \int_{\kappa Q} \int_{\kappa Q} |g(x, y)| dx dy \right)^t \leq c_{n,r,p,q} \kappa^n \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} |g(x, y)|^s dx dy \right)^{t/s}$$

for every  $g \in L^s(\mathbb{R}^n \times \mathbb{R}^n)$ , where  $s = p/r$ ,  $t = q/r$  and  $\beta = t/s = q/p$ .

*Proof.* The fractional maximal function  $\mathcal{M}_\sigma$  and the Riesz potential  $I_\sigma$  are both associated with  $\mathbb{R}^d$ . Throughout this proof  $d = 2n$  and  $\sigma = 2n(\beta - 1)/t$ .

Let us rewrite the left hand side of inequality (2.7) as

$$\begin{aligned} LHS &= \kappa^n \sum_{Q \in \mathcal{W}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{\kappa Q}(z) \chi_Q(w) \\ &\quad \left( \ell(\kappa Q)^{2n(\beta-1)/t} \frac{1}{|\kappa Q|} \int_{\kappa Q} \frac{1}{|\kappa Q|} \int_{\kappa Q} |g(x, y)| dx dy \right)^t dz dw. \end{aligned}$$

By (2.4) with  $(z, w) \in \kappa Q \times Q \subset \kappa Q \times \kappa Q \subset \mathbb{R}^d$  and by (2.5)

$$\begin{aligned} \kappa^{-n} LHS &\lesssim \sum_{Q \in \mathcal{W}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{\kappa Q}(z) \chi_Q(w) [\mathcal{M}_\sigma g(z, w)]^t dz dw \\ &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} [\mathcal{M}_\sigma g(z, w)]^t dz dw \\ &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} [I_\sigma |g|(z, w)]^t dz dw. \end{aligned}$$

Since  $1 < s = p/r < t = q/r < \infty$  and

$$\frac{r}{p} - \frac{r}{q} = \frac{\beta - 1}{t} = \frac{\sigma}{2n},$$

we obtain  $0 < 1/s - 1/t = \sigma/2n < 1$ . Hence, Theorem 2.3 yields the right hand side of inequality (2.7).  $\square$

### 3. A PROOF OF THEOREM 1.4

We prove a fractional Hardy-type inequality in a domain  $\Omega$  whose complement is  $\eta$ -plump.

*Proof of Theorem 1.4.* By [V, Theorem 3.52] and inequalities (2.1) we see that for every  $Q \in \mathcal{W}$  there is a closed cube  $Q^s$  in  $\mathbb{R}^n$  such that

$$Q^s \subset \mathbb{R}^n \setminus \overline{\Omega}, \quad \text{diam}(Q) = \text{diam}(Q^s), \quad \text{dist}(Q, Q^s) \leq 15\eta \text{diam}(Q).$$

We write  $Q^* := \kappa Q$  for the dilated cube of  $Q$  having the same centre as  $Q$  and side length  $\kappa \ell(Q)$ ,  $\kappa = 40\eta\sqrt{n}$ . The triangle inequality implies that  $Q^s \subset Q^*$ . Let

$$(3.1) \quad \alpha = s + n/q - n/p > 0.$$

Suppose that  $u \in W^{s,p}(\mathbb{R}^n)$  has support in  $\overline{\Omega}$ . By (2.1),

$$\int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial\Omega)^{\alpha q}} dx \leq \sum_{Q \in \mathcal{W}} \text{diam}(Q)^{-\alpha q} \int_Q |u(x) - u_{Q^s}|^q dx.$$

For a given  $Q \in \mathcal{W}$  the inclusion  $Q \subset Q^*$  yields

$$\int_Q |u(x) - u_{Q^s}|^q dx \lesssim \int_{Q^*} |u(x) - u_{Q^*}|^q dx + |Q| |u_{Q^s} - u_{Q^*}|^q$$

Since  $|Q| = |Q^s|$  and  $Q^s \subset Q^*$ , we obtain

$$\begin{aligned} |Q| |u_{Q^s} - u_{Q^*}|^q &= \int_{Q^s} |u_{Q^s} - u_{Q^*}|^q dx \\ &\lesssim \int_{Q^s} |u(x) - u_{Q^s}|^q dx + \int_{Q^*} |u(x) - u_{Q^*}|^q dx. \end{aligned}$$

Because  $0 < 1/p - 1/q < s/n$ , there is a number  $r \in (1, p)$  such that

$$(3.2) \quad \mu = n(1/p - 1/r) + s \in (0, s)$$

and  $0 < 1/r - 1/q < \mu/n$ . Application of Lemma 2.2 to the cubes  $Q^*$  and  $Q^s$  yields

$$\int_Q |u(x) - u_{Q^s}|^q dx \lesssim |Q^*|^{1+q\mu/n-q/r} \left( \int_{Q^*} \int_{Q^*} \frac{|u(x) - u(y)|^r}{|x - y|^{n+\mu r}} dy dx \right)^{q/r}.$$

Hence,

$$\begin{aligned} & \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial\Omega)^{\alpha q}} dx \\ & \lesssim \sum_{Q \in \mathcal{W}} \text{diam}(Q)^{-\alpha q} \int_Q |u(x) - u_{Q^s}|^q dx \\ & \lesssim \eta^{\alpha q} \sum_{Q \in \mathcal{W}} |Q^*|^{1+q(\mu/n-1/r-\alpha/n)} \left( \int_{Q^*} \int_{Q^*} \frac{|u(x) - u(y)|^r}{|x - y|^{n+\mu r}} dy dx \right)^{q/r} \\ & \lesssim \eta^{\alpha q} \sum_{Q \in \mathcal{W}} |Q^*|^{1+q(\mu/n+1/r-\alpha/n)} \left( \int_{Q^*} \int_{Q^*} \frac{|u(x) - u(y)|^r}{|x - y|^{n+\mu r}} dy dx \right)^{q/r}. \end{aligned}$$

Equations (3.1) and (3.2) imply that

$$1 + q(\mu/n + 1/r - \alpha/n) = 2q/p.$$

Hence, Lemma 2.6 yields

$$\begin{aligned} & \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial\Omega)^{\alpha q}} dx \\ & \lesssim \eta^{\alpha q} \sum_{Q \in \mathcal{W}} |Q^*|^{2q/p} \left( \int_{Q^*} \int_{Q^*} \frac{|u(x) - u(y)|^r}{|x - y|^{n+\mu r}} dy dx \right)^{q/r} \\ & \lesssim \eta^{\alpha q + n} \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{ps+n}} dx dy \right)^{q/p}. \end{aligned}$$

Since  $\alpha q + n = 2n + q(s - n/p)$ , the claim follows.  $\square$

#### 4. APPLICATIONS OF THEOREM 1.4

Let us begin with certain function spaces. The usual Besov space  $B_{pp}^s(\mathbb{R}^n)$  coincides with the Sobolev space  $W^{s,p}(\mathbb{R}^n)$ , [T2, pp. 6–7]. Hence, we may define

$$(4.1) \quad \begin{aligned} \tilde{B}_{pp}^s(\bar{\Omega}) &= \{u \in W^{s,p}(\mathbb{R}^n) : \text{spt } u \subset \bar{\Omega}\}, \\ \|u\|_{\tilde{B}_{pp}^s(\bar{\Omega})} &= \|u\|_{W^{s,p}(\mathbb{R}^n)}. \end{aligned}$$

The following corollary follows immediately from Theorem 1.4.

**4.2. Corollary.** *Suppose that  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , with an  $\eta$ -plump complement  $\mathbb{R}^n \setminus \Omega$ ,  $\eta \geq 1$ . Let  $s \in (0, 1)$  and  $p, q \in (1, \infty)$ . If  $0 < 1/p - 1/q < s/n$ , then*

$$\left( \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial\Omega)^{sq+n(1-q/p)}} dx \right)^{1/q} \leq c_{s,n,p,q} \eta^{2n/q+s-n/p} \|u\|_{\tilde{B}_{pp}^s(\bar{\Omega})}$$

for every  $u \in \tilde{B}_{pp}^s(\bar{\Omega})$ .

Related Hardy inequalities for a wider scale of Triebel–Lizorkin and Besov spaces  $\tilde{F}_{pq}^s(\bar{\Omega})$  and  $\tilde{B}_{pq}^s(\bar{\Omega})$ , respectively, have been considered in [T1]. The novelty in our result is that we only require the complement of  $\Omega$  in  $\mathbb{R}^n$  to be  $\eta$ -plump.

Let us study the validity of an intrinsic Hardy-type inequality. We focus on bounded Lipschitz domains and  $C^\infty$  domains in  $\mathbb{R}^n$ , [T3, p.64]. In both cases, the complement of  $\Omega$  in  $\mathbb{R}^n$  is  $\eta$ -plump for some  $\eta \geq 1$ . The following corollary applies to all  $u \in W^{s,p}(\Omega)$  but is restricted to the case  $0 < s < 1/p$ .

**4.3. Corollary.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$ ,  $n \geq 2$ . Let  $p, q \in (1, \infty)$  and  $s \in (0, 1/p)$ . If  $0 < 1/p - 1/q < s/n$ , then there is a constant  $c > 0$  such that the inequality*

$$(4.4) \quad \left( \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial\Omega)^{sq+n(1-q/p)}} dx \right)^{1/q} \leq c \|u\|_{L^p(\Omega)} + c \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{sp+n}} dx dy \right)^{1/p} = c \|u\|_{W^{s,p}(\Omega)}$$

holds for all  $u \in W^{s,p}(\Omega)$ .

*Proof.* Since  $B_{pp}^s(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$  the usual Besov space  $B_{pp}^s(\Omega)$  can be defined by

$$B_{pp}^s(\Omega) = \{f \in L^p(\Omega) : f = g|_{\Omega} \text{ for some } g \in W^{s,p}(\mathbb{R}^n)\},$$

$$\|f\|_{B_{pp}^s(\Omega)} = \inf \|g\|_{W^{s,p}(\mathbb{R}^n)},$$

where the infimum is taken over all functions  $g \in W^{s,p}(\mathbb{R}^n)$ ,  $g|_{\Omega} = f$ . In the following two identifications we assume that  $\Omega$  is a bounded Lipschitz domain. First,

$$\tilde{B}_{pp}^s(\bar{\Omega}) = B_{pp}^s(\Omega)$$

with equivalent norms, (4.1) and [T3, p. 66]. The spaces  $B_{pp}^s(\Omega)$  and  $W^{s,p}(\Omega)$  coincide and the norms are equivalent, [DS, Theorem 6.7] and

[T3, Theorem 1.118]. Inequality (4.4) is therefore a consequence of Corollary 4.2.  $\square$

The assumption  $0 < s < 1/p$  can be relaxed if we restrict the boundary behavior of functions. We state the following corollary.

**4.5. Corollary.** *Suppose that  $\Omega$  is a bounded  $C^\infty$  domain in  $\mathbb{R}^n$ ,  $n \geq 2$ . Let  $p, q \in (1, \infty)$  and  $s \in (0, 1)$ ,  $s \neq 1/p$ . If  $0 < 1/p - 1/q < s/n$ , then there is a constant  $c > 0$  such that the inequality*

$$(4.6) \quad \left( \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial\Omega)^{sq+n(1-q/p)}} dx \right)^{1/q} \leq c \|u\|_{L^p(\Omega)} + c \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{sp+n}} dx dy \right)^{1/p} = c \|u\|_{W^{s,p}(\Omega)}$$

holds for all

$$u \in W_0^{s,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{s,p}(\Omega)}.$$

*Proof.* Observe that  $\Omega$  is also a bounded Lipschitz domain in  $\mathbb{R}^n$ . Hence, reasoning as in the proof of Corollary 4.3 yields  $W^{s,p}(\Omega) = B_{pp}^s(\Omega)$  and, consequently,

$$W_0^{s,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{B_{pp}^s(\Omega)} = \overset{\circ}{B}_{pp}^s(\Omega).$$

Because  $s \neq 1/p$ ,

$$\overset{\circ}{B}_{pp}^s(\Omega) = \widetilde{B}_{pp}^s(\overline{\Omega});$$

we refer to [T3, pp. 66–67]. Inequality (4.6) follows from these facts and Corollary 4.2.  $\square$

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